Linear-Quadratic inverse eigenvalue problem and its Stepwise Algorithm for solving a certain $2n \times 2n$ nonsingular Hamiltonian Symmetric matrices

N.K Oladejo

Abstract- This paper investigate and examine various methods and techniques adopted by many researchers in solving linear-quadratic inverse eigenvalue problems and come out with a stepwise algorithm for solving the linear-quadratic inverse eigenvalue problem for a certain Hamiltonian symmetric matrices via Newton s iterative method.

Keywords: Stepwise, Algorithms, inverse, eigenvalue, symmetric, iterative, Hamiltonic.

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1. Introduction

The Linear- Quadratic inverse eigenvalue problem (LQIEP) has been studied by many authors and researchers (see survey by Chu and Golub (2006)).Recently solvability of the IEP for a class of singular Hamilton matrices has been obtained by Oduro et al. (2012). Oladejo, et.al (2014) examined an inverse eigenvalue problem for optimal linear quadratic control where they considers a linear quadratic optimal control from the perspective of a matrix inverse eigenvalue problem using Newton's method for solving the Inverse Eigenvalue problem for a class of Hamiltonian matrices in the neighborhood of a related singular matrix of rank 1 with numerical examples to illustrate the result.

Oladejo et.al (2015) investigates linear-quadratic optimal control problem (LQOCP) and its definiteness of an Inverse Eigenvalue Problem (IEP) on a Certain Hamilton Matrices which consists of both singular and non-singular symmetric matrices of rank 1 via Newton's method for solving the inverse eigenvalue problem for non-singular symmetric matrices in the neighborhood of the first type of matrices on linear-quadratic optimal control problem (LQOCP). Likewise Oladejo (2015) examined and reviewed least-square solution of linear-quadratic inverse eigenvalue problem (LQIEP) in Hamiltonian symmetric matrices using the nonsingular value $n \times n$ decomposition method where $n \times n$ are real matrix and represents its unique optimal approximation in the least square solutions.

Oladejo (2016) examined the existence and uniqueness solution of an optimal control problem via stochastic differential equation where he cited and reviewed various and the important properties and the solutions of such equations. A particular consequence was the connection with the classic partial differential equation (PDE) methods for studying diffusions, the Kolmogorov forward (Fokker-Planck) and backward equations where the Stochastic Differential Equations (SDE) is considered as an ordinary differential equations (ODE) driven by white noise and justified the connection between the Ito's integral and white noise in the case of non-random integrands (interpreted as test functions). The sequence of ODEs, driven by approximations to white noise limiting to an SDE which is very important in the stochastic modeling of physical systems and simulation of SDE on a computer was also considered.

Oladejo and Anang (2016) investigate and proposed derivation of an explicit function of Nonsingular Hamilton symmetric Matrices of Rank 1 via linear-quadratics inverse eigenvalue problem in the neighborhood of the first type of Hamilton matrices through numerical illustration and examples.

Oladejo (2017) cited and worked on numerical evaluation of nonlinear Hamiltonian symmetric matrix of Rank 1 of an inverse eigenvalue problem via Newton-Raphson method where he employed Newton-Raphson's method for solving the inverse eigenvalue problem in a class of Hamiltonian matrices in the neighborhood of a related nonsingular matrix of rank 1 with numerous numerical examples to illustrate the results. Based on these we investigate and examine various methods and

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techniques adopted by many researchers in solving linear-quadratic inverse eigenvalue problems and come out with a proposed a stepwise algorithm for solving the linear-quadratic inverse eigenvalue problem for a certain Hamiltonian symmetric matrices via Newton s iterative method.

2. Mathematical Formulation of linear-quadratic optimal control

We consider a linear-quadratic optimal control function of the form

$$Ix_i(u) = \int_{t_0}^{t_1} \frac{1}{2}\left[x(t)^T Qx(t) + u(t)^T Ru(t)\right]dt$$

(1)

Subject to differential equation

$$\dot{X}(t) = Ax(t) + Bu(t), t \in [t_0, t_1], x(t_i) = x_i$$

(2)

Where

$$A \in \Re^{n\times n}, B \in \Re^{n\times m}, Q \in \Re^{n\times n} : Q = Q^T \geq 0$$

and $R \in \Re^{m\times m} : R = R^T > 0$

Such that Q is a symmetric positive semi definite matrix and R is a symmetric positive definite matrix

Constructing the Hamiltonian equation from equation (1) and (2) yields:

$$H(p, x, u, t) = \frac{1}{2}\left[x^T Qx + u^T Ru\right] + p^T\left[Ax + Bu\right]$$

(3)

At any optimal input $u_\bullet$ and the corresponding state $x_\bullet$

Then;

$$\frac{\partial H}{\partial u}(p_\bullet(t), x_\bullet(t), u_\bullet(t), t) = 0$$

$$\Rightarrow u_\bullet(t)^T R + p_\bullet(t)^T B = 0$$

(4)

Thus, $u_\bullet(t) = -R^{-1}B^T p_\bullet(t)$

(5)

Finding the adjoint equation:

$$\left[\frac{\partial H}{\partial x}(p_\bullet(t), k_\bullet(t), u_\bullet(t), t)\right]^T$$

$$\Rightarrow \left(x_\bullet(t)^T Q + p_\bullet(t)^T A\right)^T$$

(6)

$$= -\dot{p}_\bullet(t), t \in [t_0, t_1], p_\bullet(t_1) = 0$$

(7)

Which yields;

$$\dot{p}_\bullet(t) = A^T p_\bullet(t) - Qx_\bullet(t), t \in [t_0, t_1], p_\bullet(t_1) = 0$$

(8)

Consequently;

$$\frac{d}{dt}\begin{bmatrix} x_\bullet(t) \\ p_\bullet(t) \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}\begin{bmatrix} x_\bullet(t) \\ p_\bullet(t) \end{bmatrix}, t \in [t_0, t_1], x_\bullet(t_0) = x_i$$

(9)

Equation (9) is then a linear, time variant differential equation in $(x_\bullet, p_\bullet)$

1. Inverse Eigenvalue Problem for a Singular $2n \times 2n$ Symmetric Matrix of Rank 1

We consider a singular symmetric matrix of order $2n \times 2n$
When n =2

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

We assume that the singularity is due to the row dependence relations specified below:

$$R_{i+1} = k_i R_1$$

$$\Rightarrow a_{21} = k_1 a_{11}; a_{22} = k_1 a_{12}, a_{23} = k_1 a_{13}, a_{24} = k_1 a_{14}$$

$$a_{31} = k_2 a_{11}; a_{32} = k_2 a_{12}, a_{33} = k_2 a_{13}, a_{34} = k_2 a_{14}$$

$$a_{41} = k_3 a_{11}; a_{42} = k_3 a_{12}, a_{43} = k_3 a_{13}, a_{44} = k_3 a_{14}$$

$$\Rightarrow a_{22} = k_1(a_{12}) = k_1(a_{21}) = k_1^{\ 2} a_{11}$$

$$a_{23} = k_1(a_{13}) = k_1(a_{31}) = k_1 k_2 a_{11}$$

$$a_{24} = k_1(a_{14}) = k_1(a_{41}) = k_1 k_3 a_{11}$$

$$a_{33} = k_2(a_{13}) = k_2(a_{31}) = k_2^{\ 2} a_{11}$$

$$a_{34} = k_2(a_{14}) = k_2(a_{41}) = k_2 k_3 a_{11}$$

$$a_{44} = k_3(a_{14}) = k_3(a_{41}) = k_3^{\ 2} a_{11}$$

Thus

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} 1 & k_1 & k_2 & k_3 \\ k_1 & k_1^{\ 2} & k_1 k_2 & k_1 k_3 \\ k_2 & k_1 k_2 & k_2^{\ 2} & k_2 k_3 \\ k_3 & k_1 k_3 & k_2 k_3 & k_3^{\ 2} \end{bmatrix}$$

To solve the inverse eigenvalue problem (IEP) we use the given nonzero eigenvalue of the form

$$tr(A) = \lambda = a_{11}(1 + k_1^{\ 2} + k_2^{\ 2} + k_3^{\ 2})$$

$$a_{11} = \frac{\lambda}{1 + k_1^{\ 2} + k_2^{\ 2} + k_3^{\ 2}} \quad \Rightarrow$$

$$A = \frac{\lambda}{1 + k_1^{\ 2} + k_2^{\ 2} + k_3^{\ 2}} \begin{bmatrix} 1 & k_1 & k_2 & k_3 \\ k_1 & k_1^{\ 2} & k_1 k_2 & k_1 k_3 \\ k_2 & k_1 k_2 & k_2^{\ 2} & k_2 k_3 \\ k_3 & k_1 k_3 & k_2 k_3 & k_3^{\ 2} \end{bmatrix}$$

Illustration

Given that $\lambda = 30, k_1 = 2, k_2 = 3, k_3 = 4$ b $\Rightarrow A =$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \\ 4 & 8 & 12 & 16 \end{bmatrix}$$

Hence, matrix A is a 4×4 singular symmetric matrix which has been reconstructed from the given nonzero eigenvalue and the prescribed dependence relation parameters.

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2. Inverse Eigenvalue Problem for a nonsingular $2n \times 2n$ symmetric matrix-

We construct a characteristic (Polynomial) function of the diagonal elements of the matrix

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}=\begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22}-\lambda & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33}-\lambda & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44}-\lambda \end{bmatrix}$$

In other words, we consider the function with independent variables defined on 4 selected elements of matrix A, precisely, the diagonal elements of the form:

$$f(a_{11},a_{22},a_{33},a_{44})=\lambda^2-(trA)\lambda+\det A$$

Thus, given four distinct eigenvalues $\lambda_1,\lambda_2,\lambda_3,\lambda_4$ we have the following three (4) functions with 4 independent variables being the diagonal element of A

$$f_1(a_{11},a_{22},a_{33},a_{44})=\left(\begin{array}{l} \lambda^4{}_1-(a_{11}+a_{22}+a_{33}+a_{44})\lambda^3{}_1+(a_{11}a_{22}+a_{11}a_{33}+a_{11}a_{44}+a_{22}a_{33} \\ +a_{22}a_{44}+a_{33}a_{44}+a_{34}a_{43}+a_{12}a_{21})\lambda^2{}_1-(a_{11}a_{22}a_{33}+a_{11}a_{21}a_{44}+ \\ a_{11}a_{33}a_{44}+a_{11}a_{33}a_{43}+a_{22}a_{33}a_{44}+a_{11}a_{34}a_{43}-a_{12}a_{21}a_{33}-a_{12}a_{21}a_{44} \\ -a_{13}a_{21}a_{32}+a_{14}a_{21}a_{42})\lambda_1+a_{11}a_{22}a_{33}a_{44}-a_{11}a_{22}a_{34}a_{43}-a_{12}a_{21}a_{33}a_{44} \\ -a_{12}a_{21}a_{34}a_{43}+a_{13}a_{21}a_{32}a_{44}-a_{13}a_{21}a_{34}a_{42}-a_{14}a_{21}a_{43}a_{32}+a_{14}a_{21}a_{42}a_{33} \end{array}\right)$$

$$f_2(a_{11},a_{22},a_{33},a_{44})=\left(\begin{array}{l} \lambda^4{}_2-(a_{11}+a_{22}+a_{33}+a_{44})\lambda^3{}_2+(a_{11}a_{22}+a_{11}a_{33}+a_{11}a_{44}+a_{22}a_{33} \\ +a_{22}a_{44}+a_{33}a_{44}+a_{34}a_{43}+a_{12}a_{21})\lambda^2{}_2-(a_{11}a_{22}a_{33}+a_{11}a_{21}a_{44}+ \\ a_{11}a_{33}a_{44}+a_{11}a_{33}a_{43}+a_{22}a_{33}a_{44}+a_{11}a_{34}a_{43}-a_{12}a_{21}a_{33}-a_{12}a_{21}a_{44} \\ -a_{13}a_{21}a_{32}+a_{14}a_{21}a_{42})\lambda_2+a_{11}a_{22}a_{33}a_{44}-a_{11}a_{22}a_{34}a_{43}-a_{12}a_{21}a_{33}a_{44} \\ -a_{12}a_{21}a_{34}a_{43}+a_{13}a_{21}a_{32}a_{44}-a_{13}a_{21}a_{34}a_{42}-a_{14}a_{21}a_{43}a_{32}+a_{14}a_{21}a_{42}a_{33} \end{array}\right)$$

$$f_3(a_{11},a_{22},a_{33},a_{44})=\left(\begin{array}{l} \lambda^4{}_3-(a_{11}+a_{22}+a_{33}+a_{44})\lambda^3{}_3+(a_{11}a_{22}+a_{11}a_{33}+a_{11}a_{44}+a_{22}a_{33} \\ +a_{22}a_{44}+a_{33}a_{44}+a_{34}a_{43}+a_{12}a_{21})\lambda^2{}_3-(a_{11}a_{22}a_{33}+a_{11}a_{21}a_{44}+ \\ a_{11}a_{33}a_{44}+a_{11}a_{33}a_{43}+a_{22}a_{33}a_{44}+a_{11}a_{34}a_{43}-a_{12}a_{21}a_{33}-a_{12}a_{21}a_{44} \\ -a_{13}a_{21}a_{32}+a_{14}a_{21}a_{42})\lambda_3+a_{11}a_{22}a_{33}a_{44}-a_{11}a_{22}a_{34}a_{43}-a_{12}a_{21}a_{33}a_{44} \\ -a_{12}a_{21}a_{34}a_{43}+a_{13}a_{21}a_{32}a_{44}-a_{13}a_{21}a_{34}a_{42}-a_{14}a_{21}a_{43}a_{32}+a_{14}a_{21}a_{42}a_{33} \end{array}\right)$$

$$f_4(a_{11},a_{22},a_{33},a_{44})=\left(\begin{array}{l} \lambda^4{}_4-(a_{11}+a_{22}+a_{33}+a_{44})\lambda^3{}_4+(a_{11}a_{22}+a_{11}a_{33}+a_{11}a_{44}+a_{22}a_{33} \\ +a_{22}a_{44}+a_{33}a_{44}+a_{34}a_{43}+a_{12}a_{21})\lambda^2{}_4-(a_{11}a_{22}a_{33}+a_{11}a_{21}a_{44}+ \\ a_{11}a_{33}a_{44}+a_{11}a_{33}a_{43}+a_{22}a_{33}a_{44}+a_{11}a_{34}a_{43}-a_{12}a_{21}a_{33}-a_{12}a_{21}a_{44} \\ -a_{13}a_{21}a_{32}+a_{14}a_{21}a_{42})\lambda_4+a_{11}a_{22}a_{33}a_{44}-a_{11}a_{22}a_{34}a_{43}-a_{12}a_{21}a_{33}a_{44} \\ -a_{12}a_{21}a_{34}a_{43}+a_{13}a_{21}a_{32}a_{44}-a_{13}a_{21}a_{34}a_{42}-a_{14}a_{21}a_{43}a_{32}+a_{14}a_{21}a_{42}a_{33} \end{array}\right)$$

DERIVATION OF AN EXPLICIT FORMULA FOR THE JACOBIAN IN THE $2n \times 2n$ CASE

Here we derived 16 explicit functional for the Jacobian from the above independent variables as follows

$$\frac{\partial f_1}{\partial a_{11}} = -\lambda_1^3 + \lambda_1^2(a_{22}+a_{33}+a_{44}) - \lambda_1(a_{22}a_{33}+a_{33}a_{44}+a_{22}a_{44}) \; ; \qquad \frac{\partial f_1}{\partial a_{22}} = -\lambda_1^3 + \lambda_1^2(a_{11}+a_{33}+a_{44}) - \lambda_1(a_{11}a_{33}+a_{33}a_{44}+a_{11}a_{44})$$

$$\frac{\partial f_1}{\partial a_{33}} = -\lambda_1^3 + \lambda_1^2(a_{11}+a_{22}+a_{44}) - \lambda_1(a_{11}a_{22}+a_{22}a_{44}+a_{11}a_{44}) \; ; \qquad \frac{\partial f_1}{\partial a_{44}} = -\lambda_1^3 + \lambda_1^2(a_{11}+a_{22}+a_{33}) - \lambda_1(a_{11}a_{22}+a_{22}a_{33}+a_{11}a_{33})$$

$$\frac{\partial f_2}{\partial a_{11}} = -\lambda_2^3 + \lambda_2^2(a_{22}+a_{33}+a_{44}) - \lambda_2(a_{22}a_{33}+a_{33}a_{44}+a_{22}a_{44}) \; ; \qquad \frac{\partial f_2}{\partial a_{22}} = -\lambda_2^3 + \lambda_2^2(a_{11}+a_{33}+a_{44}) - \lambda_2(a_{11}a_{33}+a_{33}a_{44}+a_{11}a_{44})$$

$$\frac{\partial f_2}{\partial a_{33}} = -\lambda_2^3 + \lambda_2^2(a_{11}+a_{22}+a_{44}) - \lambda_2(a_{11}a_{22}+a_{22}a_{44}+a_{11}a_{44}); \qquad \frac{\partial f_2}{\partial a_{44}} = -\lambda_2^3 + \lambda_2^2(a_{11}+a_{22}+a_{33}) - \lambda_2(a_{11}a_{22}+a_{22}a_{33}+a_{11}a_{33})$$

$$\frac{\partial f_3}{\partial a_{11}} = -\lambda_3^3 + \lambda_3^2(a_{22}+a_{33}+a_{44}) - \lambda_3(a_{22}a_{33}+a_{33}a_{44}+a_{22}a_{44}); \qquad \frac{\partial f_3}{\partial a_{22}} = -\lambda_3^3 + \lambda_3^2(a_{11}+a_{33}+a_{44}) - \lambda_3(a_{11}a_{33}+a_{33}a_{44}+a_{11}a_{44})$$

$$\frac{\partial f_3}{\partial a_{33}} = -\lambda_3^3 + \lambda_3^2(a_{11}+a_{22}+a_{44}) - \lambda_3(a_{11}a_{22}+a_{22}a_{44}+a_{11}a_{44}); \qquad \frac{\partial f_3}{\partial a_{44}} = -\lambda_3^3 + \lambda_3^2(a_{11}+a_{22}+a_{33}) - \lambda_3(a_{11}a_{22}+a_{22}a_{33}+a_{11}a_{33})$$

$$\frac{\partial f_4}{\partial a_{11}} = -\lambda_4^3 + \lambda_4^2(a_{22}+a_{33}+a_{44}) - \lambda_4(a_{22}a_{33}+a_{33}a_{44}+a_{22}a_{44}); \qquad \frac{\partial f_4}{\partial a_{22}} = -\lambda_4^3 + \lambda_4^2(a_{11}+a_{33}+a_{44}) - \lambda_4(a_{11}a_{33}+a_{33}a_{44}+a_{11}a_{44})$$

$$\frac{\partial f_4}{\partial a_{33}} = -\lambda_4^3 + \lambda_4^2(a_{11}+a_{22}+a_{44}) - \lambda_4(a_{11}a_{22}+a_{22}a_{44}+a_{11}a_{44}); \qquad \frac{\partial f_4}{\partial a_{44}} = -\lambda_4^3 + \lambda_4^2(a_{11}+a_{22}+a_{33}) - \lambda_4(a_{11}a_{22}+a_{22}a_{33}+a_{11}a_{33})$$

Thus, its' Jacobian derivation yields

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial a_{11}} & \frac{\partial f_1}{\partial a_{22}} & \frac{\partial f_1}{\partial a_{33}} & \frac{\partial f_1}{\partial a_{44}} \\ \frac{\partial f_2}{\partial a_{11}} & \frac{\partial f_2}{\partial a_{22}} & \frac{\partial f_2}{\partial a_{33}} & \frac{\partial f_2}{\partial a_{44}} \\ \frac{\partial f_3}{\partial a_{11}} & \frac{\partial f_3}{\partial a_{22}} & \frac{\partial f_3}{\partial a_{33}} & \frac{\partial f_3}{\partial a_{44}} \\ \frac{\partial f_4}{\partial a_{11}} & \frac{\partial f_4}{\partial a_{22}} & \frac{\partial f_4}{\partial a_{33}} & \frac{\partial f_4}{\partial a_{44}} \end{bmatrix} = \begin{bmatrix} -\lambda_1^3 + \lambda_1^2(a_{22}+a_{33}+a_{44}) - & -\lambda_1^3 + \lambda_1^2(a_{11}+a_{33}+a_{44}) - & -\lambda_1^3 + \lambda_1^2(a_{11}+a_{22}+a_{44}) - & -\lambda_1^3 + \lambda_1^2(a_{11}+a_{22}+a_{33}) - \\ \lambda_1(a_{11}a_{33}+a_{33}a_{44}+a_{11}a_{44}) & \lambda_1(a_{11}a_{33}+a_{33}a_{44}+a_{11}a_{44}) & \lambda_1(a_{11}a_{22}+a_{22}a_{44}+a_{11}a_{44}) & \lambda_1(a_{11}a_{22}+a_{22}a_{33}+a_{11}a_{33}) \\ -\lambda_3^3 + \lambda_3^2(a_{22}+a_{33}+a_{44}) - & -\lambda_2^3 + \lambda_2^2(a_{11}+a_{33}+a_{44}) - & -\lambda_2^3 + \lambda_2^2(a_{11}+a_{22}+a_{44}) - & -\lambda_2^3 + \lambda_2^2(a_{11}+a_{22}+a_{33}) - \\ \lambda_2(a_{22}a_{33}+a_{33}a_{44}+a_{22}a_{44}) & \lambda_2(a_{11}a_{33}+a_{33}a_{44}+a_{11}a_{44}) & \lambda_2(a_{11}a_{22}+a_{22}a_{44}+a_{11}a_{44}) & \lambda_2(a_{11}a_{22}+a_{22}a_{33}+a_{11}a_{33}) \\ -\lambda_3^3 + \lambda_3^2(a_{22}+a_{33}+a_{44}) - & -\lambda_3^3 + \lambda_3^2(a_{11}+a_{33}+a_{44}) - & -\lambda_3^3 + \lambda_3^2(a_{11}+a_{22}+a_{44}) - & -\lambda_3^3 + \lambda_3^2(a_{11}+a_{22}+a_{33}) - \\ \lambda_3(a_{22}a_{33}+a_{33}a_{44}+a_{22}a_{44}) & \lambda_3(a_{11}a_{33}+a_{33}a_{44}+a_{11}a_{44}) & \lambda_3(a_{11}a_{22}+a_{11}a_{44}+a_{22}a_{44}) & \lambda_3(a_{22}a_{33}+a_{11}a_{22}+a_{11}a_{33}) \\ -\lambda_4^3 + \lambda_4^2(a_{22}+a_{33}+a_{44}) - & -\lambda_4^3 + \lambda_4^2(a_{11}+a_{33}+a_{44}) - & -\lambda_4^3 + \lambda_2^2(a_{11}+a_{22}+a_{44}) - & -\lambda_4^3 + \lambda_4^2(a_{11}+a_{22}+a_{33}) - \\ \lambda_4(a_{22}a_{33}+a_{33}a_{44}+a_{22}a_{44}) & \lambda_4(a_{11}a_{33}+a_{33}a_{44}+a_{11}a_{44}) & \lambda_4(a_{11}a_{22}+a_{22}a_{44}+a_{11}a_{44}) & \lambda_4(a_{11}a_{22}+a_{22}a_{33}+a_{11}a_{33}) \end{bmatrix}$$

We estimate the determinant and the inverse of the Jacobian then we substitute the values of the *(n+1)th* iteration of the Newton's method as given by the following recursive relation

$$X^{(n+1)} = X^{(n)} - J^{-1}(X^{(n)})\underline{f}(X^{(n)})$$

3. Stepwise Algorithm for $2n \times 2n$ Nonsingular Hamiltonian Symmetric Matrices

Given two or more distinct target eigenvalues $\lambda_1, \lambda_2 .... \lambda_n$ (repeated for each)

Step 1: Determine the characteristic functions. ($\underline{f}(X^{(0)})$).

ie.

$$f_1(a_{11}, a_{22}) = \lambda_1^{\,2} - 2(trA)\lambda_1 + \det H$$
$$f_2(a_{11}, a_{22}) = \lambda_2^{\,2} - 2(trA)\lambda_2 + \det H$$

Step 2: Find the Jacobian from the function. i.e

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial a_{11}} & \frac{\partial f_1}{\partial a_{22}} \\ \frac{\partial f_2}{\partial a_{11}} & \frac{\partial f_2}{\partial a_{22}} \end{bmatrix} \Rightarrow \begin{bmatrix} -\lambda_1 + 2a_{22} & -\lambda_1 + 2a_{11} \\ -\lambda_2 + 2a_{22} & -\lambda_2 + 2a_{11} \end{bmatrix}$$

Step 3.Compute the Determinant of the matrix

Step 4.Determine the inverse of the Jacobian i. e

$$J^{-1} = \frac{1}{(\lambda_1 - \lambda_2)(a_{22} - a_{11})} \begin{bmatrix} a_{22} - \lambda_1 & a_{11} - \lambda_1 \\ a_{22} - \lambda_2 & a_{11} - \lambda_2 \end{bmatrix}$$

Step 5: Apply the Newton's iterative method in H .i.e.

$$X^{(1)} = X^{(0)} - J^{-1}(X^{(0)})\underline{f}(X^{(0)})$$

Step 6: Substitute $X^{(0)} = \begin{bmatrix} a_{11}^{(0)} \\ a_{22}^{(0)} \end{bmatrix}$ into H replacing the original diagonal element.

Using the formula for the inverse of the Jacobian matrix, we have:

$$J^{-1} = \frac{1}{(\lambda_1 - \lambda_2)(a_{22} - a_{11})} \begin{bmatrix} a_{22} - \lambda_1 & a_{11} - \lambda_1 \\ a_{22} - \lambda_2 & a_{11} - \lambda_2 \end{bmatrix}$$

$$J^{-1} = \frac{1}{(\lambda_1 - \lambda_2)(a_{22} - a_{11})} \begin{bmatrix} -2 & -2 \\ 1 & 5 \end{bmatrix} = -\frac{1}{12} \begin{bmatrix} -2 & -2 \\ 1 & 5 \end{bmatrix}$$

Substitute the values into the Newton's iterative equation. i.e. $X^{(1)} = X^{(0)} - J^{-1}(X^{(0)})\underline{f}(X^{(0)})$

$$X^{(0)} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} + \frac{1}{12} \begin{bmatrix} -2 & -2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 6 \\ -6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$X^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow A(X^1) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

Thus, we obtain the **(exact**) solution.

Step 7.Observe and determine the type of the matrix.

Numerical Examples

1. Suppose, the IEP to be solved is to determine completely the nonsingular symmetric coefficient matrix of the system

$$\dot{x}_1 = a_{11}x_1 + 2x_2$$
$$\dot{x}_2 = 2x_1 + a_{22}x_2$$

Given the eigenvalues $\lambda_1 = -1, \lambda_2 = 3$ i.e., given a target solution of the form

$$x = c_1 u e^{-t} + c_2 v e^{3t}$$

Assuming the initializing matrix $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$; $X^{(0)} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

Moreover, $A^{(0)}$ is singular with $k = 2, a_{11} = 1$

Also $\lambda = trA^{(0)} = 5$

We first compute the values of the functions at the initial point:

$$f_1(a_{11}, a_{22}) = \lambda_1^{\,2} - (a_{11} + a_{22})\lambda_1 + (a_{11}a_{22} - a_{12}^{\,2})$$
$$\Rightarrow 1 - 5(-1) + 0 = 6$$
$$f_2(a_{11}, a_{22}) = \lambda_2^{\,2} - (a_{11} + a_{22})\lambda_2 + (a_{11}a_{22} - a_{12}^{\,2})$$
$$\Rightarrow 9 - 5(3) + 0 = -6$$

$$\underline{f}(X^{(0)}) = \begin{bmatrix} 6 \\ -6 \end{bmatrix}$$

2. Given the target eigenvalues $\lambda_1 = 1, \lambda_2 = -2$
and an initial rank 1 singular matrix with
$a_{11} = 1, k_1 = 2 \Rightarrow \lambda = 10$
i.e.

$$A = \begin{bmatrix} 1 & 2 & i & 2i \\ -2 & 4 & -2i & 4i \\ i & -2i & 1 & 2 \\ 2i & 4i & -2 & 4 \end{bmatrix}$$

Using the above as initial matrix, we proceed as follows

$$f_1(a_{11}, a_{22}) = \lambda_1^2 - 2(a_{11} + a_{22})\lambda_1 + 0$$
$$\Rightarrow 1 - 10(1) + 0 = -9$$
$$f_2(a_{11}, a_{22}) = \lambda_2^2 - 2(a_{11} + a_{22})\lambda_2 + 0$$
$$\Rightarrow 4 - 10(-2) + 0 = 24$$
$$\underset{-}{f}(X^{(0)}) = \begin{bmatrix} -9 \\ 24 \end{bmatrix}$$

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial a_{11}} & \frac{\partial f_1}{\partial a_{22}} \\ \frac{\partial f_2}{\partial a_{11}} & \frac{\partial f_2}{\partial a_{22}} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -\lambda_1 + 2a_{22} & -\lambda_1 + 2a_{11} \\ -\lambda_2 + 2a_{22} & -\lambda_2 + 2a_{11} \end{bmatrix} = \begin{bmatrix} -1+8 & -1+2 \\ 2+8 & 2+2 \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ 10 & 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 3.167 & 2 & i & 2i \\ -2 & 3.667 & -2i & 4i \\ i & -2i & 3.167 & 2 \\ 2i & 4i & -2 & 3.667 \end{bmatrix}$$

Hence, the matrices is positive definite.

3. Given the target eigenvalues $\lambda_1 = -1, \lambda_2 = -2$
With an initial rank 1 singular matrix with $a_{11} = 1, k_1 = 2 \Rightarrow \lambda = 10$
i.e.

$$H = \begin{bmatrix} 1 & 2 & i & 2i \\ -2 & 4 & -2i & 4i \\ i & -2i & 1 & 2 \\ 2i & 4i & -2 & 4 \end{bmatrix}$$

To solve the IEP by Newton's method using the above as initial matrix, we proceed as follows

$$f_1(a_{11}, a_{22}) = \lambda_1^2 - 2(a_{11} + a_{22})\lambda_1 + 0 \qquad \Rightarrow 1 - 10(-1) + 0 = 11$$
$$f_2(a_{11}, a_{22}) = \lambda_2^2 - 2(a_{11} + a_{22})\lambda_2 + 0$$
$$\Rightarrow 4 - 10(-2) + 0 = 24$$

$$\underset{-}{f}(X^{(0)}) = \begin{bmatrix} 11 \\ 24 \end{bmatrix}$$

$$J^{-1} = \frac{1}{2(\lambda_1 - \lambda_2)(a_{22} - a_{11})} \begin{bmatrix} 2a_{22} - \lambda_1 & 2a_{11} - \lambda_1 \\ 2a_{22} - \lambda_2 & 2a_{11} - \lambda_2 \end{bmatrix} =$$

$$J^{-1} = \frac{1}{18} \begin{bmatrix} 7 & 1 \\ 10 & 4 \end{bmatrix}$$

Substituting into the Newton's equation

$$X^{(n+1)} = X^{(n)} - J^{-1}(X^{(0)})\underset{-}{f}(X^{(n)})$$

$$X^{(1)} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} - \frac{1}{18} \begin{bmatrix} 7 & 1 \\ 10 & 4 \end{bmatrix} \begin{bmatrix} -9 \\ 24 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} - \begin{bmatrix} -2.166 \\ 0.333 \end{bmatrix}$$

Hence,

$$(X^{(1)}) = \begin{bmatrix} 3.167 \\ 3.667 \end{bmatrix}$$

Thus the solution of IEP is given by

$$J=\begin{bmatrix}\dfrac{\partial f_1}{\partial a_{11}} & \dfrac{\partial f_1}{\partial a_{22}}\\ \dfrac{\partial f_2}{\partial a_{11}} & \dfrac{\partial f_2}{\partial a_{22}}\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}-\lambda_1+2a_{22} & -\lambda_1+2a_{11}\\ -\lambda_2+2a_{22} & -\lambda_2+2a_{11}\end{bmatrix}=\begin{bmatrix}1+8 & 1+2\\ 2+8 & 2+2\end{bmatrix}=\begin{bmatrix}9 & 3\\ 10 & 4\end{bmatrix}$$

$$J^{-1}=\frac{1}{2(\lambda_1-\lambda_2)(a_{22}-a_{11})}\begin{bmatrix}2a_{22}-\lambda_1 & 2a_{11}-\lambda_1\\ 2a_{22}-\lambda_2 & 2a_{11}-\lambda_2\end{bmatrix}=\frac{1}{2(3)(3)}\begin{bmatrix}8+1 & 2+1\\ 8+2 & 2+2\end{bmatrix}$$

$$J^{-1}=\frac{1}{18}\begin{bmatrix}9 & 3\\ 10 & 4\end{bmatrix}$$

Substituting into the Newton's equation

$$X^{(n+1)}=X^{(n)}-J^{-1}(X^{(0)})\underline{f}(X^{(n)})$$

$$X^{(1)}=\begin{bmatrix}1\\4\end{bmatrix}-\frac{1}{18}\begin{bmatrix}9 & 3\\ 10 & 4\end{bmatrix}\begin{bmatrix}11\\24\end{bmatrix}=\begin{bmatrix}1\\4\end{bmatrix}-\begin{bmatrix}1.5\\0.777\end{bmatrix}$$

Hence,

$$(X^{(1)})=\begin{bmatrix}-0.5\\-3.223\end{bmatrix}$$

Thus the solution of IEP is given by

$$H=\begin{bmatrix}-0.5 & 2 & i & 2i\\ -2 & -3.223 & -2i & 4i\\ i & -2i & -0.5 & 2\\ 2i & 4i & -2 & -3.223\end{bmatrix}$$

Hence, the matrix is Negative definite.

Conclusion

Various and many theoretical results on the solvability of the linear-quadratic inverse eigenvalue problem for Hamilton matrices was systematically reviewed and discussed in respect of the inverse eigenvalue problems (IEP) for certain singular and non-singular Hamiltonian matrices. Base on this we have successfully proposed and established a stepwise algorithm for solving the linear-quadratic inverse eigenvalue problem for a certain Hamiltonian symmetric matrices via Newton s iterative method.

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